CISC-102

Winter 2020

Week 9 Addendum

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

More Fun with Pascal's Triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k-1}$$

We can prove this result using mathematical induction.

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & \boxed{70} & & 56 & & 28 & & 8 & & 1
\end{array}
$$

Theorem: Let m be any natural number, then

$$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+n}{n} = \sum_{i=0}^{n} \binom{m+i}{i} = \binom{m+n+1}{n}$$

for all Natural numbers n.

Proof: We prove the result using induction on n.

Base: For $n = 1$,

$$\binom{m}{0} + \binom{m+1}{1} = m + 2 = \binom{m+2}{1}$$

Induction hypothesis: Assume:

$$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+k}{k} = \sum_{i=0}^{k} \binom{m+i}{i} = \binom{m+k+1}{k}$$

for a fixed natural number $k \geq 1$.

Induction Step: We want to show that the induction hypothesis implies:

$$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+k}{k} + \binom{m+k+1}{k+1} = \sum_{i=0}^{k+1} \binom{m+i}{i} = \binom{m+k+2}{k+1}$$

Thus we have:

$$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+k}{k} + \binom{m+k+1}{k+1} = \binom{m+k+1}{k} + \binom{m+k+1}{k+1}$$

$$= \binom{m+k+2}{k+1}$$

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&6&&4&&1&&&&\\
&&&1&&5&&10&&10&&5&&1&&&\\
&&1&&6&&15&&20&&15&&6&&1&&\\
&1&&7&&21&&35&&35&&21&&7&&1&\\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}
$$

The Hexagon Identity:

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

Algebraic Proof:

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} \times \frac{(n)!}{(k+1)!(n-k-1)!} \times \frac{(n+1)!}{(k)!(n-k+1)!}$$

$$\binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} = \frac{(n-1)!}{(k)!(n-k-1)!} \times \frac{(n)!}{(k-1)!(n-k+1)!} \times \frac{(n+1)!}{(k+1)!(n-k)!}$$